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NAVAL POSTGRADUATE SCHOOL
Monterey, California

A THEOREM ON GEOMETRIC PROBABILITY

AND APPLICATIONS

by

Peter C. C. Wang

20 November 1972

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Take any line ℓ in the plane and put a Poisson process of points with parameter λ on this line. At each point of the process construct an intersecting line, say ℓ_n at the nth point, by choosing the angle of intersection Θ_n between ℓ and ℓ_n from an underlying distribution $F(\Theta)$ such that the $< \Theta_n >$ are a sequence of independent and identically distributed random variables and additionally, independent of the Poisson process variables. Take any other line ℓ', the intersections of ℓ' by the lines ℓ_n form a Poisson process with parameter

$$\lambda \left\{ \int_{-\infty}^{0} F(\Theta^*)ds + \int_{0}^{\infty} [1-F(\Theta^*)]ds \right\}$$

where

$$\Theta^* = \cos^{-1} [(-s + \cos \alpha)(1 + s^2 - 2s \cos \alpha)^{-\frac{1}{2}}]$$

and α is the angle between lines ℓ and ℓ'. The above result will imply both Rényi's and Breiman's results [1], [2]. A different but simplified proof of the Rényi result is also included.

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A Theorem on Geometric Probability and Applications*

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1. Introduction

Rényi [1] has considered an interesting model of traffic flow on a divided highway which extends to infinity in one direction without traffic lights or other inhomogeneities. It is assumed that each car travels at a constant speed which is a random variable and passing is always possible without delays. Among others, Rényi has obtained some results regarding mainly the spatial distribution of cars along the highway when the temporal distribution of cars is assumed to be described by a Poisson Process. The purpose of this paper is to discuss a number of results that can be related to low density traffic flow models on an infinite highway. These models were initiated and developed principally in papers by Rényi [1], Weiss and Herman [2], Breiman [3] sometimes without specific reference to low density traffic flow, and these models are unified by Solomon and Wang [5] in a study of Non-homogeneous Poisson Fields of Random lines. Of the theorems presented in Section 2, all are known results [5] but the proofs are included for the sake of completeness. Main results are presented in Section 3. This main result will imply immediately the results reported in [5].

2. Low Density Traffic Flow Models

In this section, we shall provide a new proof of the Rényi theorems and include other results dealing with low density traffic flow. It will also be demonstrated for Rényi's model, that if the spatial distribution of cars is assumed to obey a Poisson process then the temporal distribution

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of cars (i.e., arrival times at some fixed position) is again a Possion process. This result establishes a crucial structural property of Rényi's model for low-density traffic. Rényi found it convenient to start from the stochastic process of entrance times of the cars at a fixed point on the highway. Other authors start from the spatial process of cars distributed in locations along the infinite highway at some fixed time according to some random law. The Poisson process is the assumed machinery governing the car entrance times or equivalently car positions and the speed distributions for each car are assumed to be independently and identically distributed (i.i.d.) with a common distribution $G(V)$. Starting from this spatial process, Breiman [3] considered the idealized model and proved that the Poisson process is the only process obeying the time-invariance property-namely, if at a time t_o, the spatial process is Poisson with specific parameter, and the speeds of the cars are i.i.d. with respect to each other and the positions of the cars at time t_o, then the process will have the same properties at any other time t. Similarly, we can obtain the spatial-invariance property-namely, if at a location x_0, the temporal process is Poisson with specific parameter, and the speeds of the cars are i.i.d. with respect to each other and the entrance times of the cars at location x_0, then the process will have the same properties at any other location x. These invariant properties insure that the randomness is unaffected by the choice of origin (i.e., randomness is unaffected by translation). Furthermore, the underlying process has independent stationary increments. It is obvious that the number of cars in a zero distance is zero and the number of cars arriving in a zero temporal interval is zero. In order to establish that the underlying process is Poisson, it is now sufficient to show that the number of cars in any temporal (or spatial)

interval has a Poisson distribution [7, p. 119].

In detail, the assumptions of Renyi's model are:

(i) Instants $< t_i >_{i=1}^{\infty}$ at which cars enter the highway at a fixed position form a homogeneous Poisson process with parameter ω.

(ii) A car arriving at a certain point on the highway at instant t_i chooses a velocity V_i and then moves with this constant velocity. The random variables $< V_k >$ are independently and identically distributed with distribution function $G(v) = \Pr\{ V \leq v \}$ and sequences $< V_k >$ and $< t_k >$ are independent.

(iii) $\int_0^{\infty} \frac{1}{v}\, dG(v) < \infty$ i.e., the mean value of $\frac{1}{v}$ is finite; without this condition a traffic jam would arise and make all traffic flow impossible.

(iv) No delay in overtaking a car traveling at a slower speed when it is approached.

Suppose an arbitrary car $K(t_o, v_o)$ arrives at some fixed point of the highway at time t_o where it assumes and maintains the fixed speed v_o. Let $< t_k^+ >$ ($< t_k^- >$) denote the instants at which the car $K(t_o, v_o)$ is overtaken by faster (overtaking slower) cars. Rényi has obtained the following results.

Theorem 1. (Rényi) The instants $< t_k^+ >$ and $< t_k^- >$ form two independent homogeneous Poisson processes, with parameters:

$$\omega^+(v_o) = \omega \int_0^{v_o} \frac{v_o - v}{v}\, dG(v) \quad \text{and} \quad \omega^-(v_o) = \omega \int_{v_o}^{\infty} \frac{v - v_o}{v}\, dG(v).$$

Rényi's proof of the above theorem is based on the following two properties of Poisson processes:

(A) If $< t_i >$ are the instants of time when an event occurs in a homogeneous Poisson process with parameter ω, and $\zeta_1, \zeta_2, \ldots$ is a sequence

of independent positive random variables, each having the same distribution function $G(\zeta)$ and each independent of the process $< t_k >$, then the time instants $t_k \zeta_k$ $(k = 1,2...)$ also form a homogeneous Poisson process with density

$$\omega^* = \omega \int_0^\infty \frac{1}{\zeta} \, dG(\zeta).$$

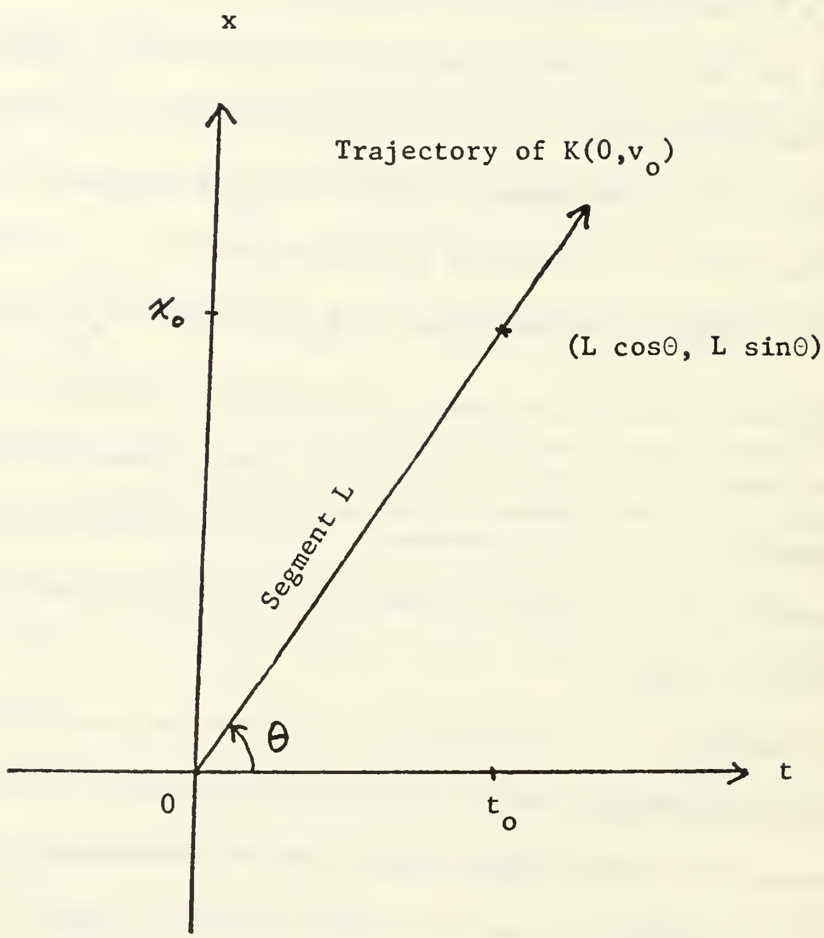
(B) If a subsequence $< t_{v_k} >$ of the instants $< t_k >$, in which an event occurs in a Poisson process with density ω, is selected at random in such a way that for each j the probability of the event A_j that j should belong to the subsequence $< v_k >$ is equal to $r (0 < r < 1)$ and the events $A_j (j = 1,2,...)$ are independent; and if $< t_{u_k} >$ are the instants that are not selected, (i.e., j belongs to the sequence $< u_k >$ if and only if it does not belong to the sequence $< v_k >$), then $< t_{v_j} >$ and $< t_{u_k} >$ are two independent Poisson processes with density ωr and $\omega(1-r)$.

It is now known from a result of Wang [6] that property (B) is a characteristic property for Poisson processes. We shall establish Theorem 1 without using property (A). It can be shown that property B implies property A (see Theorem 2 below).

Proof of Theorem 1. The trajectory of any car in time-space (Diagram 1) for Renyi's low density traffic model is realized by a straight line. Let us denote the trajectories of all cars on the highway as a set A. Denote M_L^+ the number of lines in A that intersect segment L of the trajectory of car $K(0,v_o)$ from below and M_p the number of lines in A whose arrival times are in $(0,t_o)$. Then

Diagram 1.

Time-Space Diagram



$$\Pr\{M_L^+ = n\} = \sum_{m=n}^{\infty} \binom{m}{n} \mu^n (1-\mu)^{m-n} e^{-\omega t_o} \frac{(\omega t_o)^m}{m!}$$

$$= e^{-\omega t_o \mu} \frac{(\omega t_o \mu)^n}{n!}$$

where

$$\mu = \Pr\{M_L^+ = 1 \mid M_p = 1\}$$

$$= \frac{1}{t_o} \int_0^{t_o} \Pr\{V \geq \frac{x_o}{t_o - p}\} \, dp$$

$$= \int_{v_o}^{\infty} \frac{v - v_o}{v} \, dG(v), \text{ where } v_o = \frac{x_o}{t_o} .$$

Similarly, we define M_L^- as the number of lines in A intersecting L from above and M_p^c as the number of lines in A whose arrival times fall in the interval $-c (c > 0)$ and 0. We can compute

$$\Pr\{M_L^- = n\} = \lim_{c \to \infty} e^{-\omega c \mu_c^*} \frac{(\omega c \mu_c^*)^n}{n!}$$

where

$$\mu_c^* = \Pr\{M_L^- = 1 \mid M_p^c = 1\} .$$

$$\lim_{c \to \infty} c\mu_c^* = \lim_{c \to \infty} c \Pr\{M_L^- = 1 \mid M_p^c = 1\}$$

$$= \lim_{c \to \infty} \int_{-c}^{0} \int_{0}^{\frac{x_o}{t_o - p}} dG(v)dp$$

$$= \int_{-\infty}^{0} \int_{0}^{\frac{x_o}{t_o - p}} dG(v)dp$$

$$= \int_{0}^{v_o} \frac{x_o - vt_o}{v} dG(v)$$

$$= t_o \int_{0}^{v_o} \frac{v_o - v}{v} dG(v) \quad .$$

Random variables M_L^- and M_L^+ are independent because the events involved come from disjoint intervals. This completes the proof of Theorem 1. The counting interval employed in the above theorem is on the time axis. In what follows, a similar approach to the problem dealing with a spatial counting interval is employed and produces some interesting results. Denote M^- the number of lines in A intersecting L from above and $M_{x_o}^c$ for the number of lines in A whose spatial positions at $t = 0$ are between 0 and $-c$. Let λ^* be the spatial density.

Then we have

$$\Pr\{M^- = n\} = e^{-\lambda^* x_o \mu} \frac{(\lambda^* x_o \mu)^n}{n!}$$

where

$$\mu = \Pr\{M^- = 1 \mid M_{x_o} = 1\}$$

$$= \frac{t_o}{x_o} \int_0^{v_o} (v_o - v)\,dG(v) \;;$$

and

$$\Pr\{M^+ = n\} = \lim_{c \to \infty} \sum_{m=n}^{\infty} \Pr\{M^+ = n \mid M^c_{x_o} = m\} \Pr\{M^c_{x_o} = m\}$$

$$= \lim_{c \to \infty} e^{-\lambda^* c \mu_1^c} \frac{(\lambda^* c \mu_1^c)^n}{n!}$$

where

$$\mu_1^c = \Pr\{M^+ = 1 \mid M^c_{x_o} = 1\} \;.$$

It can be easily verified that

$$\lim_{c \to \infty} c\mu_1^c = t_o \int_{v_o}^{\infty} (v - v_o)\,dG(v) \;,$$

and random variables M^- and M^+ are independent. Now denote $M = M^+ + M^-$ and we conclude that

$$\Pr\{M = n\} = e^{-\lambda^* t_o \int_o^{\infty} |v_o - v|\,dG(v)} \cdot \frac{[\lambda^* t_o \int_o^{\infty} |v_o - v|\,dG(v)]^n}{n!} \;.$$

The above result appeared initially in a paper by Weiss and Herman[2] who arrived at it from different considerations.

In the next paragraph, results are stated about the spatial distribution of vehicles if the temporal process is assumed to be Poisson.

Denote S^+ the number of lines in A intersecting $(0, x_o)$ and $x_o > 0$ at time zero and S_c^+ the number of lines in A whose arrival times are in the interval $(-c, 0)$. We further denote S^- the number of lines in A intersec-

ting $(-x_o,0)$, $x_o > 0$ at time zero and S_c^- the number of lines in A whose arrival times are in the interval $(0,c)$. Let us compute quantities: $\Pr\{S^+ = n\}$, $\Pr\{S^- = n\}$ and $\Pr\{S = S^+ S^- = n\}$.

$$\Pr\{S^+ = n\} = \lim_{c \to \infty} e^{-\omega c \mu_2} \frac{(\omega c \mu_2)^n}{n!}$$

where ω is the temporal density and

$$\mu_2 = \Pr\{S^+ = 1 \mid S_c^+ = 1\} .$$

It can be shown easily that

$$\lim_{c \to \infty} c\mu_2 = x_o \int_0^\infty \frac{1}{v}\, dG(v).$$

We conclude that

$$\Pr\{S^+ = n\} = e^{-\omega x_o \int_o^\infty \frac{1}{v} dG(v)} \frac{[\omega x_o \int_o^\infty \frac{1}{v}\, dG(v)]^n}{n!} .$$

Similarly, we obtain

$$\Pr\{S^- = n\} = \Pr\{S^+ = n\} ,$$

and

$$\Pr\{S = S^+ + S^- = n\} = e^{-2\omega x_o \int_o^\infty \frac{1}{v} dG(v)} \cdot \frac{[2\omega x_o \int_o^\infty \frac{1}{v}\, dG(v)]^n}{n!} .$$

We can now summarize as follows.

Theorem 2. If $< t_i >$ forms a Poisson process with parameter ω and sequences $< t_i >$ and $< V_i >$ are independent then the locations of vehicles on the highway at time $t = 0$ namely $< x_i >$ forms a Poisson process with parameter

$$\omega \int_0^\infty \frac{1}{v}\, dG(v)$$

Theorem 3. If $< x_i >$ forms a Poisson process with parameter λ^* and sequences $< x_i >$ and $< V_i >$ are independent and $< x_i^+ >$ denotes the positions at which the car $K(0,v_o)$ overtakes slower cars and $< x_i^- >$ denotes the positions at which car $K(0,v_o)$ is overtaken by faster cars; then the two sequences $< x_i^+ >$ and $< x_i^- >$ form two independent (homogeneous) Poisson processes, with parameters:

$$\lambda_+^*(v_o) = \lambda^* \int_0^{v_o} (v_o - v)dG(v) \quad \text{and} \quad \lambda_-^*(v_o) = \lambda^* \int_{v_o}^{\infty} (v - v_o)dG(v) \quad .$$

This result is analogous to the Rényi result which we developed as Theorem 1 except that the counting of overtakings is accomplished on the spatial axis rather than on the time axis. The next theorem provides a result analogous to that in Theorem 2.

Theorem 4. If $< x_i >$ forms a Poisson process with parameter λ^* and sequences $< x_i >$ and $< V_i >$ are independent and $< V_i >$ are i.i.d. random variables with common distribution $G(\cdot) = \Pr\{V \le v\}$; then the corresponding $< t_i >$ arrival times at position $x = 0$ forms a Poisson process with parameter $\lambda^* E(V)$ where

$$E(V) = \int_0^{\infty} v dG(v).$$

Proof. The proof is again based on the binomial mixing as presented in Property (B) and hence details are omitted.

3. Main Results

It is the purpose of this section to establish a result (Theorem 5 below) that by applying property (B) to it would imply both Rényi's and Breiman's results.

Let us take any line ℓ in the plane and put a Poisson process of points

on this line. At each point of the process construct an intersecting line ℓ_n at the n-th point S_n by chosing the angle of intersection Θ_n between lines ℓ and ℓ_n from an underlying distribution $F(\Theta)$ such that the $< \Theta_n >$ are a sequence of independent and identically distributed random variables and additionally, independent of the Poisson process. Now, take any other line ℓ' and look at the intersections of line ℓ' by the lines ℓ_n. We wish to show that the intersection points $<S_n'>$ form a Poisson process and the sequence $< \Theta_n' >$ of the consecutive angles of intersection is independent and identically distributed. It is clear that if the line ℓ' is parallel to the line ℓ, then the intersection points $< S_n' >$ of lines ℓ_n and ℓ' is again Poisson with the same parameter as the process on the line ℓ because of the invariance property mentioned in section 1. We can now state the following.

<u>Theorem</u> 5. If $< S_n >$ forms a Poisson process with parameter λ on line ℓ and sequences $< \Theta_n >$ are i.i.d. with a common distribution $F(\Theta)$, $0 \leq \Theta < \Pi$, and furthermore, the sequences $< S_n >$ and $< \Theta_n >$ are indenpendent , then

(i) the sequence $< S_n' >$ forms a Poisson process on line ℓ' with parameter

$$\lambda^* = \lambda\{\int_{-\infty}^{0} F(\Theta^*)ds_n + \int_{0}^{\infty} [1-F(\Theta^*)]ds_n\}$$

where

$$\Theta^* = \cos^{-1} [(-s_n + \cos\alpha)\ (1 + s_n^2 - 2s_n\cos\alpha)^{-\frac{1}{2}}]$$

and α is the angle between lines ℓ and ℓ', and

(ii) the angles $< \Theta_n' >$ formed between lines ℓ' and ℓ_n are i.i.d. with common distribution $F(\Theta' + \alpha)$, $- \alpha \leq \Theta' < \Pi - \alpha$.

Since the counting measure on line ℓ is invariant under translation and the

corresponding process has independent stationary increments, it is sufficient to show that the sequences $< S_n' >$ forms a Poisson process by showing that the counting measure has a Poisson distribution. i.e., to show that

$$\Pr\{N_{\ell'} = k\} = e^{-\lambda^*} \frac{(\lambda^*)^k}{k!} \; ; \; k = 0, 1, 2, \ldots \; .$$

where the random variable N_{ℓ}' denotes the number of intersections between lines ℓ_n and ℓ' in a given unit interval on line ℓ'.

Proof of Theorem 5.

Because of the invariance property mentioned in section one, it is sufficient to choose the intersection of lines ℓ' and ℓ as the origin and denote the random variable N_{ℓ}^c the number of points in the interval $(-c,c)$ on line ℓ. Then we have

$$\Pr\{N_{\ell'} = k\} = \lim_{c \to \infty} \sum_{m=k}^{\infty} \Pr\{N_{\ell'} = k | N_{\ell}^c = m\} \Pr\{N_{\ell}^c = m\}$$

$$= \lim_{c \to \infty} e^{-2c\lambda\mu} \frac{(2c\lambda\mu)^k}{k!} \; ; \; k = 0, 1, 2, \ldots,$$

where

$$\mu = \Pr\{N_{\ell'} = 1 | N_{\ell}^c = 1\}.$$

Upon evaluation of μ we have

$$\lim_{c \to \infty} 2c\mu = \int_{-\infty}^{0} F(\Theta^*) dS_n + \int_{0}^{\infty} [1-F(\Theta^*)] dS_n \quad .$$

Furthermore, the angles $< \Theta_n' > \; = \; < \Theta_n - \alpha >$ (see diagram 2) are clearly i.i.d. with common distribution $F(\Theta' + \alpha)$ for $- \alpha \leq \Theta < \Pi - \alpha$. This completes the proof of the theorem.

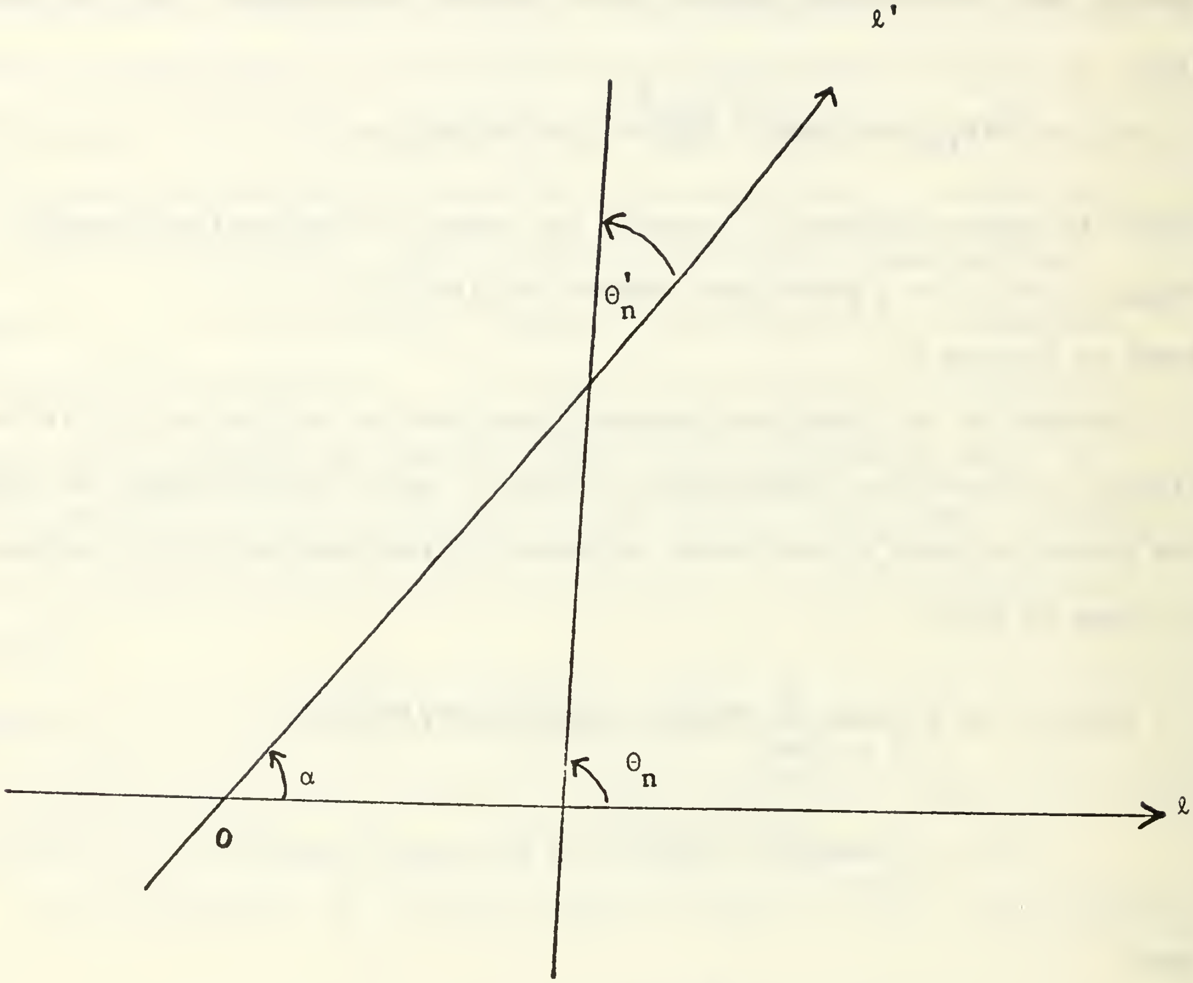


Diagram 2

It is reasonable to conjecture that the sequences $< \Theta_n' >$ and $< S_n' >$ formed in Theorem 5 are independent sequences, however it is not true in general unless the lines $< \ell_n >$ are all parallel lines. To see this, we can use the following indirect argument:

Based on the results stated in Theorem 2 and Theorem 4 plus the independence condition needed, one might expect to get the following identity

$$\omega = \omega \int_0^\infty \frac{1}{v}\, dG(v) \int_0^\infty v\, dG(v)$$

But the identity is true if and only if all cars are travelling at the same speed.

Applying property B to Theorem 5 and restricting $\alpha = \Pi/2$ and $0 \leq \Theta < \frac{\Pi}{2}$ will result Rényi's Theorem. Other theorems stated in section 2 are implied by Theorem 5 in a similar manner.

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13. ABSTRACT

Take any line ℓ in the plane and put a Poisson process of points with parameter λ on this line. At each point of the process construct an intersecting line, say ℓ_n at the nth point, by choosing the angle of intersection Θ_n between ℓ and ℓ_n from an underlying distribution $F(\Theta)$ such that the $< \Theta_n >$ are a sequence of independent and identically distributed random variables and additionally, independent of the Poisson process variables. Take any other line ℓ', the intersections of ℓ' by the lines ℓ_n form a Poisson process with parameter

$$\lambda \left\{ \int_{-\infty}^{0} F(\Theta^*)ds + \int_{0}^{\infty} [1-F(\Theta^*)]ds \right\}$$

where

$$\Theta^* = \cos^{-1} [(-s + \cos\alpha)(1 + s^2 - 2s\cos\alpha)^{-\frac{1}{2}}]$$

and α is the angle between lines ℓ and ℓ'. The above result will imply both Rényi's and Breiman's results [1], [2]. A different but simplified proof of the Rényi result is also included.

14 KEY WORDS	LINK A		LINK B		LINK C	
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